NIMO Summer Contest 2011 Solutions

1. A jar contains 4 blue marbles, 3 green marbles, and 5 red marbles. If Helen reaches in the jar and selects a marble at random, then the probability that she selects a red marble can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

   Answer: $\boxed{17}$ Exactly 5 out of the 12 possible outcomes give Helen a red marble, so the probability that she selects a red marble is $\frac{5}{12}$. So $m+n=17$.

2. The sum of three consecutive integers is 15. Determine their product.

   Answer: $\boxed{120}$ If the middle integer is x, then $(x-1)+x+(x+1)=15$ or $x=5$. Hence, the desired product is $4\cdot 5\cdot 6=120$.

3. Define $\lfloor x\rfloor$ as the largest integer less than or equal to x. Define $\{x\}=x-\lfloor x\rfloor$. For example, $\{3\}=3-3=0$, $\{\pi\}=\pi-3$, and $\{-\pi\}=4-\pi$. If $\{n\}+\{3n\}=1.4$, then find the sum of all possible values of $100\{n\}$.

   Answer: $\boxed{145}$ Note that $\{3n\}$ must equal $3\{n\}$, $3\{n\}-1$, or $3\{n\}-2$.

   If $\{3n\}=3\{n\}$, then $4\{n\}=1.4 \implies \{n\}=0.35$, which does not satisfy $\{3n\}=3\{n\}$.
   If $\{3n\}=3\{n\}-1$, then $4\{n\}=2.4 \implies \{n\}=0.6$, which is a solution.
   If $\{3n\}=3\{n\}-2$, then $4\{n\}=3.4 \implies \{n\}=0.85$, which is a solution.

   Hence, the sum of all possible values of $100\{n\}$ is $60+85=145$.

4. Find the number of ordered pairs of integers (a,b) that satisfy the inequality

$$1<a<b+2<10.$$

   Answer: $\boxed{28}$ Note that $a\in\{2,3,\ldots,8\}$. For each a, there are $9-a$ solutions for b; namely, $a-1,a,a+1,\ldots,7$. Thus, there are $7+6+\cdots+1=28$ solutions.

5. In equilateral triangle ABC, the midpoint of $\overline{BC}$ is M. If the circumcircle of triangle MAB has area 36π, then find the perimeter of the triangle.

   Answer: $\boxed{36}$ Because $\triangle ABC$ is equilateral, $\measuredangle AMB=90^\circ$. Hence, $\overline{AB}$ is a diameter of the circumcircle of $\triangle MAB$, so $AB=12$. It follows that the perimeter of the triangle is 36.

6. If the answer to this problem is x, then compute the value of $\frac{x^2}{8}+2$.

   Answer: $\boxed{4}$ x must satisfy the equation $x=\frac{x^2}{8}+2$, which rearranges to $(x-4)^2=0$. So the only possible value for x is 4.

7. Let $P(x)=x^2-20x-11$. If a and b are natural numbers such that a is composite, $\gcd(a,b)=1$, and $P(a)=P(b)$, compute ab.

   Note: $\gcd(m,n)$ denotes the greatest common divisor of m and n.

   Answer: $\boxed{99}$ Note that $P(a)-P(b)=(a+b-20)(a-b)=0$. Hence, $a=b$ or $a+b=20$. Clearly the first case is absurd, so we need only check the case $a+b=20$. It is easy to verify that the only solutions are $(9,11)$ and $(11,9)$, so $ab=99$.

8. Triangle ABC with $\measuredangle A=90^\circ$ has incenter I. A circle passing through A with center I is drawn, intersecting $\overline{BC}$ at E and F such that $BE<BF$. If $\frac{BE}{EF}=\frac{2}{3}$, then $\frac{CF}{FE}=\frac{m}{n}$, where m and n are

relatively prime positive integers. Find $m+n$.

Answer: $\boxed{7}$ Denote by X and Y the second intersections of the circle with $\overline{AB}$ and $\overline{AC}$, respectively. Denote by P, Q, and R the projections of I onto $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively.

Because I is the incenter of the triangle, $IR = IP$. Additionally, $BI = BI$, and $\angle BRI \cong \angle BPI$ by definition. Hence, $\triangle BIR \cong \triangle BIP$. We may similarly conclude that $\triangle IRA \cong \triangle IPF$ (using the fact that $IA = IF$) and $\triangle XIR \cong \triangle EIP$ (using the fact that $IX = IE$). It follows that $BR = BP$, $RA = PF$, and $XR = EP$, or $BX = BE$ and $BA = BF$. This yields $AX = EF$. Similarly, $AX = AY = EF$ and $CF = CY$.

Let $BE = BX = 2x$, so $AX = AY = EF = 3x$. Let $CY = CF = y$. Then, by the Pythagorean Theorem,

$$\begin{aligned} AB^2 + AC^2 &= BC^2 \\ (5x)^2 + (3x+y)^2 &= (5x+y)^2 \\ 25x^2 + 9x^2 + 6xy + y^2 &= 25x^2 + 10xy + y^2 \\ 4xy &= 9x^2 \\ y &= \tfrac{9}{4}x. \end{aligned}$$

It follows that $\frac{CF}{FE} = \frac{9}{4}x \cdot \frac{1}{3x} = \frac{3}{4}$, so $m+n = 7$.

9. The roots of the polynomial $P(x) = x^3 + 5x + 4$ are r, s, and t. Evaluate $(r+s)^4(s+t)^4(t+r)^4$.

Answer: $\boxed{256}$ By Vieta's Formulas, $r+s+t = 0$ and $rst = -4$. Hence,

$$\begin{aligned} (r+s)^4(s+t)^4(t+r)^4 &= (-t)^4(-r)^4(-s)^4 \\ &= (rst)^4 \\ &= 256. \end{aligned}$$

10. The edges and diagonals of convex pentagon $ABCDE$ are all colored either red or blue. How many ways are there to color the segments such that there is exactly one monochromatic triangle with vertices among A, B, C, D, E; that is, triangles whose edges are all the same color?

Answer: $\boxed{260}$ Assume, WLOG, that ABC is the monochromatic triangle. Furthermore, assume that $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ are colored red. We multiply by $2\binom{5}{2} = 20$ later to compensate.

If $\overline{DE}$ is colored blue, then either $\overline{AD}$ or $\overline{AE}$ is red. Assume $\overline{AE}$ is red. From $\triangle ACE$, we deduce that $\overline{CE}$ is blue. Hence, $\overline{CD}$ is red. But we have a contradiction with $\triangle BDE$, so there are no solutions in this case.

If $\overline{DE}$ is colored red, then the remaining segments to be colored are $\overline{AD}$, $\overline{BD}$, $\overline{CD}$, $\overline{AE}$, $\overline{BE}$, and $\overline{CE}$. It is easy to confirm that the possible sets of these segments to be colored red are $\{\}$, $\{\overline{AD}\}$, $\{\overline{BD}\}$, $\{\overline{CD}\}$, $\{\overline{AE}\}$, $\{\overline{BE}\}$, $\{\overline{CE}\}$, $\{\overline{AD}, \overline{BE}\}$, $\{\overline{AD}, \overline{CE}\}$, $\{\overline{BD}, \overline{AE}\}$, $\{\overline{BD}, \overline{CE}\}$, $\{\overline{CD}, \overline{AE}\}$, or $\{\overline{CD}, \overline{BE}\}$, for a total of 13 sets.

Finally, we multiply by 20, for the final answer of $13 \times 20 = 260$.

11. How many ordered pairs of positive integers (m, n) satisfy the system

$$\begin{aligned} \gcd(m^3, n^2) &= 2^2 \cdot 3^2, \\ \text{LCM}[m^2, n^3] &= 2^4 \cdot 3^4 \cdot 5^6, \end{aligned}$$

where $\gcd(a, b)$ and $\text{LCM}[a, b]$ denote the greatest common divisor and least common multiple of a and b, respectively?

Answer: $\boxed{2}$ Clearly, the prime factors of m and n are limited to 2, 3, and 5. Denote $m = 2^{\alpha_1}3^{\alpha_2}5^{\alpha_3}$ and $n = 2^{\beta_1}3^{\beta_2}5^{\beta_3}$. Then, we may write

$$\gcd(m^3, n^2) = 2^{\min(3\alpha_1, 2\beta_1)}3^{\min(3\alpha_2, 2\beta_2)}5^{\min(3\alpha_3, 2\beta_3)}$$
$$\text{LCM}[m^2, n^3] = 2^{\max(2\alpha_1, 3\beta_1)}3^{\max(2\alpha_2, 3\beta_2)}5^{\max(2\alpha_3, 3\beta_3)}.$$

It follows that $\min(3\alpha_1, 2\beta_1) = 2$. But α_1 and β_1 are whole numbers, so by parity it follows that $\beta_1 = 1$. Similarly, $\max(2\alpha_1, 3\beta_1) = 4$, so $\alpha_1 = 2$.

We may similarly conclude that $\beta_2 = 1$ and $\alpha_2 = 2$. Now, $\min(3\alpha_3, 2\beta_3) = 0$ and $\max(2\alpha_3, 3\beta_3) = 6$, so $(\alpha_3, \beta_3) = (3, 0)$ or $(0, 2)$. These correspond to the solutions $(m, n) = (4500, 6), (36, 60)$, for a total of 2 solutions.

12. In triangle ABC, $AB = 100$, $BC = 120$, and $CA = 140$. Points D and F lie on $\overline{BC}$ and $\overline{AB}$, respectively, such that $BD = 90$ and $AF = 60$. Point E is an arbitrary point on $\overline{AC}$. Denote the intersection of $\overline{BE}$ and $\overline{CF}$ as K, the intersection of $\overline{AD}$ and $\overline{CF}$ as L, and the intersection of $\overline{AD}$ and $\overline{BE}$ as M. If $[KLM] = [AME]+[BKF]+[CLD]$, where $[X]$ denotes the area of region X, compute CE.

Answer: $\boxed{91}$ Note that

$$\begin{aligned}[CAD] + [ABE] + [BCF] &= [ABC] + ([AME] + [BKF] + [CLD] - [KLM]) \\ &= [ABC].\end{aligned}$$

Additionally, $[CAD] = \frac{30}{120}[ABC]$, $[ABE] = \frac{AE}{AC}[ABC]$, and $[BCF] = \frac{40}{100}[ABC]$. Substituting yields $\frac{1}{4} + \frac{AE}{AC} + \frac{2}{5} = 1$, or $\frac{AE}{AC} = \frac{7}{20}$. It follows that $AE = 49$, and thus $CE = 140 - 49 = 91$.

13. For real θ_i, $i = 1, 2, \ldots, 2011$, find the maximum value of the expression

$$\sin^{2012}\theta_1\cos^{2012}\theta_2 + \sin^{2012}\theta_2\cos^{2012}\theta_3 + \cdots + \sin^{2012}\theta_{2010}\cos^{2012}\theta_{2011} + \sin^{2012}\theta_{2011}\cos^{2012}\theta_1.$$

Answer: $\boxed{1005}$ Let S be the expression given in the problem statement. We have:

$$S \le \sin^2\theta_1\cos^2\theta_2 + \sin^2\theta_2\cos^2\theta_3 + \cdots + \sin^2\theta_{2010}\cos^2\theta_{2011} + \sin^2\theta_{2011} + \cos^2\theta_1.$$

Now, let $\sin^2\theta_i = x_i$, so $\cos^2\theta_i = 1 - x_i$. Then

$$S \le x_1(1 - x_2) + x_2(1 - x_3) + x_3(1 - x_4) + \cdots + x_{2011}(1 - x_1).$$

Consider some i. All of the terms in the sum involving x_i sum to $x_i(1 - x_{i+1} - x_{i-1})$, where indices are taken modulo 2011, so the maximum value of the expression occurs when $x_i = 0$ or 1. If all of the x_i's are equal to 0, then the sum is equal to 0. Otherwise, we may assume WLOG that $x_1 = 1$. Then, $x_2 + x_{2011} \ne 2$, so we may assume $x_2 = 0$. It follows that $x_3 = 1$, $x_4 = 0$, $x_5 = 1, \ldots, x_{2011} = 1$. Thus,

$$\begin{aligned}x_1(1 - x_2) + x_2(1 - x_3) + x_3(1 - x_4) + \cdots + x_{2011}(1 - x_1) &\le (1 + 0) + (1 + 0) + \cdots + (1 + 0) + 1 \\ &= 1005.\end{aligned}$$

Hence, $S \le 1005$. But equality holds in every inequality when we use the x_i's described above, so the maximum value for S is 1005.

14. In circle ω_1 with radius 1, circles $\phi_1, \phi_2, \ldots, \phi_8$, with equal radii, are drawn such that for $1 \le i \le 8$, ϕ_i is tangent to ω_1, ϕ_{i-1}, and ϕ_{i+1}, where $\phi_0 = \phi_8$ and $\phi_1 = \phi_9$. There exists a circle ω_2 such that $\omega_1 \ne \omega_2$ and ω_2 is tangent to ϕ_i for $1 \le i \le 8$. The radius of ω_2 can be expressed in the form $a - b\sqrt{c} - d\sqrt{e - \sqrt{f}} + g\sqrt{h - j\sqrt{k}}$ such that $a, b, \ldots, j$ are positive integers and the numbers

$c, f, k, \gcd(h, j)$ are squarefree. What is $a+b+c+d+e+f+g+h+j+k$?

Answer: $\boxed{31}$ Let O be the center of ω_1. By symmetry, O is also the center of ω_2. Let A be the intersection of ϕ_1 and ω_1, let B be the center of ϕ_1, let C be the intersection of $\overleftrightarrow{OB}$ and ω_1, and let D be the intersection of ϕ_1 and ϕ_2. Let $AB = BD = BC = r$, so $OB = 1-r$. But $\angle BOD = \frac{\pi}{8}$ and $\angle ODB = \frac{\pi}{2}$, so $\sin\frac{\pi}{8} = \frac{r}{1-r}$. By the half-angle formula, $\frac{\sqrt{2-\sqrt{2}}}{2} = \frac{r}{1-r}$. Solving, we find $r = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2-\sqrt{2}}+2}$. Thus, the radius of ω_2 is $1-2r = -1+\frac{4}{\sqrt{2-\sqrt{2}}+2} = 7-4\sqrt{2}-4\sqrt{2-\sqrt{2}}+2\sqrt{4-2\sqrt{2}}$, so $a+b+\cdots+k = 31$.

15. Let

$$N = \sum_{a_1=0}^{2}\sum_{a_2=0}^{a_1}\sum_{a_3=0}^{a_2}\cdots\sum_{a_{2011}=0}^{a_{2010}}\left[\prod_{n=1}^{2011} a_n\right].$$

Find the remainder when N is divided by 1000.

Answer: $\boxed{95}$ If $a_i = 0$ for some i, then $\prod_{n=1}^{2011} a_n = 0$, so we need not consider this case. Thus, $a_i = 1$ or 2 for all i.

Because $a_i \geq a_{i+1}$, it follows that $a_1 = a_2 = \cdots = a_j = 2$ and $a_{j+1} = a_{j+2} = \cdots = a_{2011} = 1$ for some j, or all of the a_i's are equal. For each $j = 1, 2, \ldots, 2010$, this choice of a_i's adds a total of 2^j to the sum. The cases where all of the a_i's are equal add 1 and 2^{2011} to the sum. Hence,

$$N = 1 + 2 + 2^2 + \cdots + 2^{2011} = 2^{2012} - 1.$$

It remains to reduce N modulo 1000. Note that

$$2^{2012} \equiv 0 \pmod{8},$$
$$2^{2012} \equiv \left(2^{100}\right)^{20} \cdot 2^{12} \equiv 2^{12} \pmod{125},$$

by Euler's Theorem. By the Chinese Remainder Theorem, $2^{2012} \equiv 2^{12} \equiv 96 \pmod{1000}$. Hence, $N \equiv 96 - 1 = 95 \pmod{1000}$.