NIMO (NEE-mo) Summer Contest 2011

1. A jar contains 4 blue marbles, 3 green marbles, and 5 red marbles. If Helen reaches in the jar and selects a marble at random, then the probability that she selects a red marble can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

2. The sum of three consecutive integers is 15. Determine their product.

3. Define $\lfloor x \rfloor$ as the largest integer less than or equal to x. Define $\{x\} = x - \lfloor x \rfloor$. For example, $\{3\} = 3 - 3 = 0$, $\{\pi\} = \pi - 3$, and $\{-\pi\} = 4 - \pi$. If $\{n\} + \{3n\} = 1.4$, then find the sum of all possible values of $100\{n\}$.

4. Find the number of ordered pairs of integers (a, b) that satisfy the inequality
$$1 < a < b + 2 < 10.$$

5. In equilateral triangle ABC, the midpoint of $\overline{BC}$ is M. If the circumcircle of triangle MAB has area 36π, then find the perimeter of the triangle.

6. If the answer to this problem is x, then compute the value of $\frac{x^2}{8} + 2$.

7. Let $P(x) = x^2 - 20x - 11$. If a and b are natural numbers such that a is composite, $\gcd(a, b) = 1$, and $P(a) = P(b)$, compute ab.

   Note: $\gcd(m, n)$ denotes the greatest common divisor of m and n.

8. Triangle ABC with $\measuredangle A = 90^\circ$ has incenter I. A circle passing through A with center I is drawn, intersecting $\overline{BC}$ at E and F such that $BE < BF$. If $\frac{BE}{EF} = \frac{2}{3}$, then $\frac{CF}{FE} = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

9. The roots of the polynomial $P(x) = x^3 + 5x + 4$ are r, s, and t. Evaluate $(r+s)^4(s+t)^4(t+r)^4$.

10. The edges and diagonals of convex pentagon $ABCDE$ are all colored either red or blue. How many ways are there to color the segments such that there is exactly one monochromatic triangle with vertices among A, B, C, D, E; that is, triangles whose edges are all the same color?

11. How many ordered pairs of positive integers (m, n) satisfy the system
$$\gcd(m^3, n^2) = 2^2 \cdot 3^2,$$
$$\text{LCM}[m^2, n^3] = 2^4 \cdot 3^4 \cdot 5^6,$$
where $\gcd(a, b)$ and $\text{LCM}[a, b]$ denote the greatest common divisor and least common multiple of a and b, respectively?

12. In triangle ABC, $AB = 100$, $BC = 120$, and $CA = 140$. Points D and F lie on $\overline{BC}$ and $\overline{AB}$, respectively, such that $BD = 90$ and $AF = 60$. Point E is an arbitrary point on $\overline{AC}$. Denote the intersection of $\overline{BE}$ and $\overline{CF}$ as K, the intersection of $\overline{AD}$ and $\overline{CF}$ as L, and the intersection of $\overline{AD}$ and $\overline{BE}$ as M. If $[KLM] = [AME] + [BKF] + [CLD]$, where $[X]$ denotes the area of region X, compute CE.

13. For real θ_i, $i = 1, 2, \dots, 2011$, find the maximum value of the expression
$$\sin^{2012}\theta_1 \cos^{2012}\theta_2 + \sin^{2012}\theta_2 \cos^{2012}\theta_3 + \cdots + \sin^{2012}\theta_{2010} \cos^{2012}\theta_{2011} + \sin^{2012}\theta_{2011} \cos^{2012}\theta_1.$$

14. In circle ω_1 with radius 1, circles $\phi_1, \phi_2, \dots, \phi_8$, with equal radii, are drawn such that for $1 \le i \le 8$, ϕ_i is tangent to ω_1, ϕ_{i-1}, and ϕ_{i+1}, where $\phi_0 = \phi_8$ and $\phi_1 = \phi_9$. There exists a circle ω_2 such that $\omega_1 \ne \omega_2$ and ω_2 is tangent to ϕ_i for $1 \le i \le 8$. The radius of ω_2 can be expressed in the form $a - b\sqrt{c} - d\sqrt{e - \sqrt{f}} + g\sqrt{h - j\sqrt{k}}$ such that $a, b, \dots, k$ are positive integers and the numbers $c, f, k, \gcd(h, j)$ are squarefree. What is $a + b + c + d + e + f + g + h + j + k$?

15. Let

$$N = \sum_{a_1=0}^{2} \sum_{a_2=0}^{a_1} \sum_{a_3=0}^{a_2} \cdots \sum_{a_{2011}=0}^{a_{2010}} \left[\prod_{n=1}^{2011} a_n \right].$$

Find the remainder when N is divided by 1000.